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NATURAL WAVENUMBERS OF ACOUSTIC AND ELECTROMAGNETIC OSCILLATIONS IN THE VICINITY OF A CIRCULAR CASCADE WITH A CORE

V. L. Okulov

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For eigenvalue problems in diffraction theory the square of the wavenumber is usually adopted as the characteristic ("natural") parameter [1]. Rigorous and approximate methods are fairly well known for determining the natural wavenumbers (natural frequencies) of inner problems, but only the long-wave or short-wave approximations are considered for the most part in outer problems. The author is aware of only a few papers in which the eigenvalue problem has been solved in a rigorous setting (see, e.g., [2, 3] and the bibliographies therein). In the present article we determine the natural wavenumbers of the outer problem of the diffraction of electromagnetic or acoustic waves by a plane circular cascade with a core (hub) in a rigorous setting, i.e., for arbitrary ratios of the cascade period to the wavelength.

Circular cascades are customarily used to model the impellers or rotors of centrifugal compressors and fans. The solution of the eigenvalue problem should be useful in analyzing the acoustic resonance effect occurring in certain operating regimes of these machines [2]. Structures analogous to circular cascades can be regarded as models of electrodynamic resonators, certain antennas, and waveguide devices. To predict their resonance properties it is also necessary to know the natural wave numbers of electromagnetic oscillations in the vicinity of similar open structures [1].

1. We consider a stationary plane circular blade cascade of diameter 2R formed by N thin radial blades (reflectors) attached to a circular core of radius r (Fig. 1). Let the function $\varphi(\rho, \theta)$ describe the wave amplitude of steady-state acoustic or electromagnetic oscillations in the exterior of the cascade (ρ, θ are polar coordinates with origin at the center of the cascade). The amplitude of the total field can be written in the form

$$\varphi = \sum_{l=0}^{N-1} \varphi_l,$$

where each component φ_l of the total field satisfies:

the homogeneous Helmholtz equation

$$(\Delta + k^2)\varphi_l = 0,$$

where k is an arbitrary complex number;

the homogeneous Dirichlet ($\Omega = 0$) or Neumann ($\Omega = 1$) conditions

$$\varphi_l = 0 \quad \text{or} \quad \partial\varphi_l/\partial\mathbf{n} = 0$$

on the blades forming the cascade and on the surface of the core;

the generalized radiation condition [3]

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$$\varphi_l = \sum_{s=-\infty}^{\infty} a_s H_s^{(1)}(k\rho)\exp(is\theta) \quad \text{for} \quad \rho > R,$$

where $H_s^{(1)}(k\rho)$ is a Hankel function of the first kind, with $0 \leq \arg k < 2\pi$; the condition of generalized periodicity with respect to θ

$$\varphi_l(\rho, \theta + \alpha) = \varphi_l(\rho, \theta)\exp(i\mu),$$

where $\alpha = 2\pi/N$ is the θ spacing of the cascade and $\mu = \alpha l$ characterizes the phase shift between the oscillations in the interblade channels;

the condition of bounded energy at the sharp tips of the blades and at $\rho = 0$ in the case of a degenerate core ($r = 0$) [4].

We interpret the natural wavenumbers of the stated problem as those values of k for which there exists a nontrivial solution φ_l satisfying all the enumerated conditions. We note that the results of [4] are valid for the stated problem. Consequently, the set of natural wavenumbers on the complex plane is discrete, and all of them (except zero) lie below the real axis.

2. We represent the solution of the problem in the form

$$\varphi_l = \begin{cases} \sum\limits_{\nu=Ns+l} a_\nu \xi^\nu \dfrac{H_\nu^{(1)}(k\rho)}{H_\nu^{(1)}(kR)}, & \rho \geqslant R, \\ \sum\limits_{2\nu=Ns} b_{2\nu}\xi_m^{l-\nu}\xi^\nu \dfrac{Z_\nu(k\rho)}{Z_\nu(kR)}, & r \leqslant \rho \leqslant R, \quad 0 \leqslant \theta - \alpha m < \alpha, \end{cases} \tag{2.1}$$

$$s = 0, \pm 1, \pm 2, \ldots, m = 0, 1, 2, \ldots, N-1,$$

where $b_{Ns} = (-1)^\Omega b_{-Ns}$; $\xi = \exp(i\theta)$; $\xi_m = \exp(i\alpha m)$, and the function Z_ν is given by the expression

$$Z_\nu(k\rho) = \begin{cases} J_\nu(k\rho), & r = 0, \\ N_\nu(kr)J_\nu(k\rho) - J_\nu(kr)N_\nu(k\rho), & r > 0, \quad \Omega = 0, \\ N'_\nu(kr)J_\nu(k\rho) - J'_\nu(kr)N_\nu(k\rho), & r > 0, \quad \Omega = 1, \end{cases}$$

where $J_\nu(x)$ and $N_\nu(x)$ are Bessel and Neumann functions and the prime signifies the derivative with respect to ρ. By virtue of the finite-energy condition at the tips of the blades $\{a_n\}$ and $\{b_n\}$ in (2.1) belong to the class of l^2-sequences. We note that in the case of a degenerate core expression (2.1) coincides exactly with the representation of the secondary field in the problem discussed in [5] concerning a fan of thin strips.

By virtue of the generalized periodicity property it is sufficient to match the solution (2.1) at $\rho = R$ for $0 \leq \theta < \alpha$. As a result, we obtain

$$\sum_{\nu=Ns+l} a_\nu \xi^\nu = \sum_{2\nu=Ns} b_{2\nu}\xi^\nu, \tag{2.2}$$

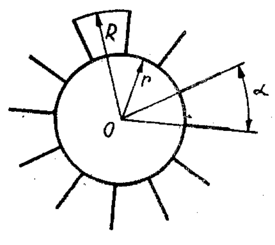

Fig. 1

$$\sum_{\nu=Ns+l} a_\nu \xi^\nu \zeta_\nu^H = \sum_{2\nu=Ns} b_{2\nu}\xi^\nu \zeta_\nu^Z, \quad 0 \leqslant \theta < \alpha,$$

$$\zeta_\nu^H = kH_\nu^{(1)\prime}(kR)/H_\nu^{(1)}(kR), \quad \zeta_\nu^Z = kZ_\nu'(kR)/Z_\nu(kR).$$

Multiplying (2.2) by $\exp\{-i(Ns + l)\theta\}$ and integrating with respect to θ from 0 to α, we obtain an infinite system of algebraic equations, which can be written as follows after the elimination of a_ν and certain identity transformations:

$$\sum_{2\nu=Ns} B_{2\nu} \frac{\zeta_\sigma^H - \zeta_\nu^Z}{\sigma^2 - \nu^2} = 0, \quad \sigma = Ns + l, \quad s = 0, \pm 1, \pm 2, \ldots, \tag{2.3}$$

where $B_{2\nu} = b_{2\nu}(1 - (-1)^s \xi_1^l)$. We note that the system (2.3) is obtained with the following restrictions on the parameters involved on it: The cases $l = 0$ and $2l = N$ are disregarded (they will be analyzed below); the values of k coinciding with the roots of the equations $H_{Ns+l}^{(1)}(kR) = 0$ and $Z_{Ns/2}(kR) = 0$ are eliminated. The roots of these equations determine the natural wavenumbers of the oscillations in the exterior of a cylinder of radius R and in the interior of an annular sector. Inasmuch as the former is possible only in the limit $r \to R$ and the eigenvalues of the annular sector are real, we can eliminate small neighborhoods of these quantities.

We have thus reduced the stated problem to the determination of those k for which the homogeneous infinite system (2.3) has a nontrivial solution.

3. We consider the cases $l = 0$ and $2l = N$. After matching the solution (2.1) on the arc $\rho = R$, $0 \leqslant \theta < \alpha$ we have for $l = 0$

$$\sum_{\nu=Ns} a_\nu \xi^\nu = \sum_{2\nu=Ns} b_{2\nu}\xi^\nu, \quad \sum_{\nu=Ns} a_\nu \xi^\nu \zeta_\nu^H = \sum_{2\nu=Ns} b_{2\nu}\xi^\nu \zeta_\nu^Z \tag{3.1}$$

and for $2l = N$

$$\sum_{2\nu=N(2s+1)} a_\nu \xi^\nu = \sum_{2\nu=Ns} b_{2\nu}\xi^\nu, \quad \sum_{2\nu=N(2s+1)} a_\nu \xi^\nu \zeta_\nu^H = \sum_{2\nu=Ns} b_{2\nu}\xi^\nu \zeta_\nu^Z. \tag{3.2}$$

Proceeding as in the derivation of (2.3), we obtain the following systems of equations from (3.1) and (3.2):

for $l = 0$

$$\frac{c}{\sigma} b_{2\sigma}(\zeta_\sigma^H - \zeta_\sigma^Z) \pm \sum_{2\nu=N(2n+1)} b_{2\nu} \frac{\zeta_\sigma^H - \zeta_\nu^Z}{\sigma^2 - \nu^2} = 0, \quad \sigma = Nn, \ n = 0, 1, 2\ldots; \tag{3.3}$$

for $2l = N$

$$\frac{c}{\sigma} b_{2\sigma}(\zeta_\sigma^H - \zeta_\sigma^Z) \pm \sum_{\nu=Nn} b_{2\nu} \frac{\zeta_\sigma^H - \zeta_\nu^Z}{\sigma^2 - \nu^2} = 0, \quad 2\sigma = N(2n+1), \tag{3.4}$$

$$n = 0, 1, 2, \ldots, \quad c = \text{const}.$$

Adding and subtracting the equations for identical σ, we infer from (3.3) and (3.4) that for $l = 0$

$$\frac{1}{\sigma} b_{2\sigma}(\zeta_\sigma^H - \zeta_\sigma^Z) = 0; \tag{3.5a}$$

$$\sum_{2\nu=N(2n+1)} b_{2\nu} \frac{\zeta_\sigma^H - \zeta_\nu^Z}{\sigma^2 - \nu^2} = 0, \quad \sigma = Nn, \quad n = 0, 1, 2, \ldots; \tag{3.5b}$$

for $2l = N$

$$\frac{1}{\sigma} b_{2\sigma} (\zeta_\sigma^H - \zeta_\sigma^Z) = 0; \tag{3.6a}$$

$$\sum_{\nu=Nn} b_{2\nu} \frac{\zeta_\sigma^H - \zeta_\nu^Z}{\sigma^2 - \nu^2} = 0, \quad 2\sigma = N(2n+1), \quad n = 0, 1, 2, \ldots \tag{3.6b}$$

It is evident from Eqs. (3.5a) and (3.6a) that nonzero amplitude coefficients in the expansion (2.1) can exist when the expression in the parentheses vanishes. To determine the conditions for this to be possible we note that

$$\zeta_\sigma^H - \zeta_\sigma^Z = \begin{cases} ic_0 & \text{for} \quad r = 0, \\ c_0 H_\sigma^{(1)}(kr) & \text{for} \quad r > 0, \ \Omega = 0, \\ c_0 H_\sigma^{(1)\prime}(kr) & \text{for} \quad r > 0, \ \Omega = 1, \end{cases} \tag{3.7}$$

where $c_0 = -2(\pi kRH_\sigma^{(1)}(kR)Z_\sigma(kR))^{-1}$. Analyzing (3.7), we infer that: a) For a degenerate core (r = 0) and finite k the coefficients b_n in Eqs. (3.5a) and (3.6a) can only be zero; b) for r > 0 there can exist finite natural wavenumbers that coincide with the corresponding wave numbers of free oscillations around a cylinder of radius r.

Consequently, for phase shifts $\mu = 0$ and $\mu = \pi$ it is possible to have free oscillations that "do not notice" the blades of the cascade. This is explained by the fact that an integral number of half-waves of the external field fits exactly between the blades in the azimuth direction (this number is even for $\mu = 0$ and odd for $\mu = \pi$). The different free-oscillation modes in the given situations are determined by the values of the parameter k for which the infinite systems (3.5b), (3.6b) have a nontrivial solution.

4. The eigenvalues of the system (2.3), (3.5b), and (3.6b) can be determined by the reduction method. To show this, we use the semiinversion method [5] to reduce the investigated systems to systems of equations of type 2. They can be written in the matrix form

$$L(k)\mathbf{x} = \mathbf{x} + T(k)\mathbf{x} = 0, \ \mathbf{x} \in l^2.$$

Invoking the asymptotic representations for the cylinder functions [6] and the estimates given in [5], we establish by direct verification the analyticity of the operator-valued function T(k) and the compactness of the operator T for all k. Also, the Fredholm analytical theorem [7] implies the existence and analyticity of an operator-valued function $L^{-1}(k)$ on the entire complex plane with the exception of a discrete set of eigenvalues. Then, in accordance with [8], the eigenvalues of the reduced systems converge to the eigenvalues of the exact systems as their order is increased. On the other hand, each eigenvalue of the exact system is the limit of the eigenvalues of the corresponding reduced systems.

The principal difficulty encountered in determining the matrix elements is associated with the computation of the cylinder functions involved in the ratios ζ_ν^H and ζ_ν^Z, because the Neumann function grows without bound and the Bessel function decreases with increasing ν. To surmount this difficulty we represent ζ_ν^H and ζ_ν^Z in terms of the ratios of cylinder functions:

$$\zeta_\nu^J = \frac{J_\nu'(x)}{J_\nu(x)}, \quad \zeta_\nu^N = \frac{N_\nu'(x)}{N_\nu(x)}, \quad \zeta_\nu^{J/N} = \frac{J_\nu(x)}{N_\nu(x)}.$$

We use recursion formulas to compute these ratios:

for ζ_ν^N in the forward direction

$$\zeta_\nu^N = -\frac{\nu}{x} + \left(\frac{\nu-1}{x} - \zeta_{\nu-1}^N\right)^{-1}, \tag{4.1}$$

and for ζ_ν^J in the reverse direction

$$\zeta_\nu^J = \frac{\nu}{x} - \left(\frac{\nu+1}{x} + \zeta_{\nu+1}^J\right)^{-1} \tag{4.2}$$

(the directions are relative to increasing ν). The computation of $\zeta_\nu^{J/N}$ is reduced by means of the relation

TABLE 1

ν	k_ν^0	$\mathrm{Re}\,k_\nu$	$\mathrm{Im}\,k_\nu$	ν	k_ν^0	$\mathrm{Re}\,k_\nu$	$\mathrm{Im}\,k_\nu$
1	2,40048	2,2395	—0,00004	4	11,79153	10,68	—0,41
2	5,52007	5,117	—0,0004	5	14,93091	13,49	—0,96
3	8,65372	7,897	—0,004				

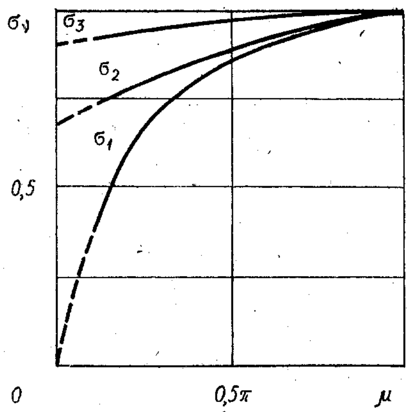

Fig. 2

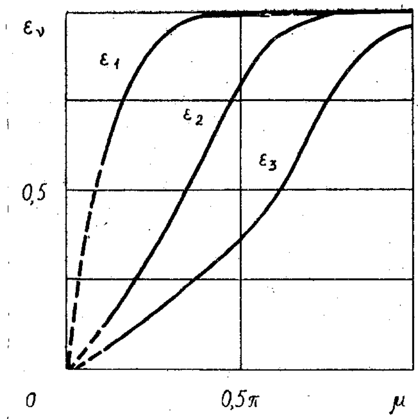

Fig. 3

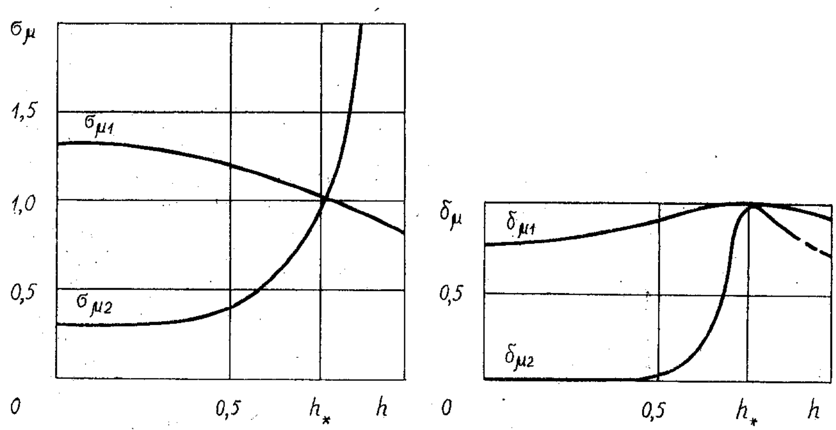

Fig. 4

$$\zeta_{\nu+1}^{J/N} = \zeta_\nu^{J/N}\left(\frac{\nu}{x} - \zeta_\nu^J\right)\Big/\left(\frac{\nu}{x} - \zeta_\nu^N\right) \tag{4.3}$$

to the functions already determined. The initial functions (4.1) and (4.3) are determined by representing the zeroth- and first-order cylinder functions in series form [6], and for (4.2) we use the expression

$$\zeta_M^J = \frac{M}{x} - \gamma(M, x),$$

where $\gamma(M, x)$ is the decomposition of the ratio $J_M(x)/J_{M-1}(x)$ into a continued fraction [6]. For $\gamma(M, x)$ there is an upper bound on the error of approximation of this continued fraction by a finite segment $\gamma_L(M, x)$:

$$|\gamma(M, x) - \gamma_L(M, x)| \leqslant \left(\frac{|x|}{2}\right)^{L-M-1} \frac{M!\,(M+1)!}{(M+L+1)!}.$$

The technique described here for computing ζ_ν^H and ζ_ν^Z permits the domain of application of the calculated parameters of the problem to be expanded considerably.

5. The results of a numerical calculation of the natural wavenumbers for a 20-blade cascade (for R = 1, Ω = 1) are summarized in Table 1 and shown in Figs. 2-4. The systems

(2.3), (3.5b), and (3.6b) were truncated to 50 equations for the calculations. The rate of convergence was tested by varying the number of equations of the truncated systems. It was confirmed that doubling the number of equations improves the precision of the natural wavenumber in the fifth significant figure.

To analyze the results we introduce the critical wavenumbers k°, which we define as the natural wavenumbers of free-oscillation modes in the interblade domains bounded by arcs of circles enclosing the cascade. In the given situation the k° coincide with the roots of the equations

$$Z_{\nu}(k^{0}R)=0,\ 2\nu=Nn,\ n=0,1,2,\ldots$$

The first five natural wavenumbers are given in Table 1 for a cascade with a degenerate core for $\mu = \pi$, along with, for comparison, the nearest critical values, which are the roots of the equation $J_0(k^{\circ}_{\nu}) = 0$ [6].

Figure 2 shows the variation of the quantity $\sigma_{\nu} = \mathrm{Re}\, k_{\nu}(\mu, 0)/\mathrm{Re}\, k_{\nu}(\pi, 0)$, and Fig. 3 the variation of $\varepsilon_{\nu} = \exp\{\mathrm{Im}\, k_{\nu}(\mu, 0)\}$ as a function of the phase shift μ for the first three natural wavenumbers. We note that their minimum deviation from the real axis corresponds to the value of the parameter $\mu = \pi$. This means that high-Q oscillations of resonators and the most pronounced acoustic resonance are possible for oscillations with the opposite phase in adjacent interblade domains. This conclusion does not contradict the results of the theory of plane infinite straight cascades [2, 9].

Figure 4 shows the real and imaginary parts of the first natural wavenumber as a function of the hub ratio $h = r/R$ for different phase shifts $\mu_1 = 0.1\pi$ and $\mu_2 = \pi$. The quantities referred to their values for $h_* = 0.761$ are plotted along the vertical axis:

$$\sigma_{\mu}=\mathrm{Re}\,k_0(\mu,h)/\mathrm{Re}\,k_0(\mu,h_*);\quad \delta_{\mu}=\mathrm{Im}\,k_0(\mu,h)/\mathrm{Im}\,k_0(\mu,h_*),$$

where $k_0(\mu_1, h_*) = 0.645 - i0.685$ and $k_0(\mu_2, h_*) = 6.85 - i0.45$. It is important to note the difference in the behavior of the real part for different values of μ as $h \to 1$. Here, as in the case of straight cascades [2], two limiting cases can be discerned:

1) $\mu = \pi$, when the natural wavenumbers are closest to their critical values and grow without bound together with them as $h \to 1$;

2) $\mu \to 0$, when the natural wavenumbers approach the values governing the free-oscillation modes of the space surrounding the cascade and tend in the limit ($h \to 1$) to the natural wavenumbers of the oscillations around a cylinder of radius R.

In particular, for $\mu = 0.1\pi$ the natural wavenumber limit coincides with the value $k = 0.5012 - i0.6435$, which is the natural wavenumber of the oscillations around a cylinder of radius $R = 1$ [3].

The maximum deviation of the natural wavenumbers from the real axis when the characteristic azimuthal and radial dimensions of the cascade coincide ($h = h_*$) is evidently explained by the existence of a transition regime from free oscillations of one direction to oscillations of the other direction. Analyzing the data in [10], we can conclude that a similar situation occurs for straight cascades.

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LARGE-PARTICLE STUDY OF THE FLOW AROUND WORKING BLADES IN A STEAM TURBINE

Yu. M. Davydov, V. D. Kulikov, and E. V. Maiorskii UDC 621.165-226.1.001.5

High supersonic velocities and a very complicated flow structure occur in the flow around the blade profiles in steam turbines, particularly the peripheral sections of the latter stages in low-pressure cylinders. It is therefore impossible to predict details of the flow. Measurements on such blades are complicated and expensive. It is therefore desirable to use numerical simulation in a preliminary analysis of the flow structure.

For example, calculations have been performed by Godunov's method [1] on the working and nozzle profiles [2-4].

The large-particle method is now widely used [5, 6], particularly for many aspects of gas dynamics, including the calculation of internal flows [7]. Here we demonstrate its use in numerical examination of a new class of topics: calculating the flow around turbine blade profiles.

Figure 1 shows one of the sets of blades (form I). The calculations were performed for an inlet angle $\beta_1 = 163°$, an angle of attack $i = \beta_1 - \beta_0 = \pm 4°13'$, and a relative pitch of $\bar{t} = t/b = 1.02$.

We considered typical modes of flow around the blades, in which subsonic velocities $M_1 \simeq 0.5$ occurred at the inlet and supersonic ones $M_2 \simeq 1.9$.

The working region ACDE (Fig. 1) of rectangular shape is split up into several zones differing in the sizes of the rectangular cells in the immobile net. The smallest cells lie in the regions of the inlet edges F and the outlet ones K, where the curvature is maximal. Here the profile contour was calculated with the necessary accuracy.

The total number of cells varied from 4000 to 6000. The calculations were performed with an ES-1040 computer (OS operating system) on a FORTRAN program; the run time for one model was not more than 6 h.

The boundary conditions were specified as follows. At the boundaries AC and ED, periodicity conditions applied. The boundary AE (Fig. 1) was taken at a distance t along the normal to the input front (line al). Test calculations showed that any further increase in this distance had no effect on the results. The conditions for constancy of the entropy S, total enthalpy J_0, and direction of the velocity vector β_1 were taken at the boundary AE:

$$S = p/\rho^k = \text{const}, \quad J_0 = \frac{k}{k-1}\frac{p}{\rho} + \frac{W^2}{2} = \text{const}, \quad \beta_1 = \text{const},$$

where p, ρ, W, and k are correspondingly pressure, density, velocity, and isentrope parameters. Also we maintained the condition for conservation of the left Riemann invariant in each time layer [8]: $RM = W - 2a(k - 1)$, where $a = \sqrt{kp/\rho}$.

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